THE ELASTIC ARCH.

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It is usual in the discussion of the mathematical principles of the inelastic arch, as an arch constructed of masonry is, in effect, to trace the curve of pressures due to the loading and to the thrust of the arch, and compare it with the configuration of the arch itself. This comparison shows, in the clearest manner, the stability or instability of the arch, and it enables the designer to form a ready opinion as to the effect of altering the shape of the arch, or of changing the loading, etc.

This curve of pressures is also, as is well known, the curve of equilibrium or catenary due to the loading and to the thrust. It is also frequently spoken of as the curve of moments, since it is well known that its ordinates multipled by the horizontal thrust are equal to the bending moments which would be caused by the given loading in an elastic girder on which no horizontal thrust acts, which girder is not necessarily straight

It is thus seen that the girder itself can be discussed by the help of an equilibrium polygon having any assumed thrust; and this is the process employed in treating the girder by the graphical method.

Now when we turn to the mathematical treatment of the elastic arch, or curved girder of any shape, acted on by a thrust, (which is the case when the reactions of its two supports are not parallel), we find that the discussions heretofore given have been based on analytic considerations exclusively, and the useful relations expressed by the equilibrium polygon are left out of view.

It is the object of this paper to point out the fundamental relationship existing between the elastic arch or crooked girder and the equilibrium curve due to its loading and to the thrust acting upon it. This relationship constitutes the basis of a complete graphical treatment of the elastic arch, subject to any possible conditions such as induced bending moments at its extremities or elsewhere, or the introduction of hinge joints at arbitrary points; and it affords at the same time a simple conception and interpretation of the analytic results arrived at in the usual investigation of the elastic arch.

It is evident that the bending moment at any point of an elastic arch is the algebraic sum of the moments of the forces and couples applied to the arch on either side of the assumed point.

Let us first consider the bending moment at any point of the arch which is due to a simple thrust alone. A simple thrust is induced in an arch by a variation of temperature or any other cause by which its natural span is made to differ from the actual distance between its points of support; but the bending moments due to a given thrust are identical whatever causes that thrust; whether it is a secondary effect due to the bending which the loading produces, or whether it is caused by some variation of temperature or position of its extremities which makes the natural span of the arch differ from the actual distance between its points of support. The word thrust is used to include the effects of contraction as well as elongation in the arch; the thrust in the former case being negative.

If the arch is not fixed in direction at the points of support, then the bending moment vanishes at these points, and the thrust acts along a line joining them, which is not necessarily horizontal. The bending moment at any point of the arch due to this thrust is the product of the thrust by its arm, which arm is the perpendicular distance from the assumed point to the line of the thrust. This product is evidently equal to the product of the horizontal component of the thrust by the vertical distance from the assumed point to the line of thrusts, as appears from an elementary consideration of the similarity of triangles.

Hence it appears that we have in this case arrived at the following important truth:

The neutral axis of an elastic arch is the equilibrium curve of the bending moments due to the thrust between its supports.

And this statement applies not only to an arch having hinge joints at its points of support, but to the elastic arch in general, as we now proceed to show.

Any case other than that already considered may be caused by the application of a couple at one or both of the points of support, thereby inducing a bending moment at one or both of these points. The effect of such a couple is not dependent upon the manner in which it is induced: it may be that it is arbitrarily applied, or it may be caused, as is usually the case, by the thrust: the bending moment, which a couple arbitrarily applied at a point of support causes, is one which uniformly decreases from its point of application to the other point of support. Its effect is then to remove the thrust line from the point where the couple is applied to a new position, such that the vertical distance between the point of support, at which the couple is applied, and the new thrust line, multiplied by the thrust of the arch, is equal to the moment of the applied couple. The other extremity of the thrust line is unmoved by applying this couple. If in addition a second couple be applied at the second point of support the thrust line is removed from that point of support in a similar manner while the other extremity is unmoved.

But the same reasoning now applies, which we before employed, to find the moment at any point of the arch; it may be stated more explicitly thus:

The bending moment at any point of an elastic arch caused by the thrust, horizontal or inclined, is the product of the horizontal component of the thrust by the vertical ordinate between the point assumed on the neutral axis of the arch and the thrust line in its true position.

It may be noticed in this connection that it is always possible to determine the magnitude of the couples accompanying a simple thrust and applied at the points of support, from consideration of the conditions imposed on the arch by the amount of deflection horizontal or otherwise which is possible, but it does not fall within the scope of this paper to examine these conditions and show how to determine the couples accompanying a thrust.

Having considered the thrust and its equilibrium or moment curve, which is the neutral axis of the arch itself, let us in the second place consider the loading and its moment curve.

The bending moment at any assumed point of the arch due to the weights is the algebraic sum of the products obtained by multiplying each weight by its horizontal distance from the assumed point.

If the arch has hinge joints at the points of support, the loading can cause no bending moments at those points; but, if the arch is supported in some other way, it is evident that the moments due to the loading are accompanied by couples applied at the points of support, in the same manner as were the bending moments due to the thrust, and that the magnitude of these couples must be determined from the same considerations respecting deflection, etc., as determined those; indeed, each separate force which is applied to the arch, be it thrust, weight or any other force, causes bending moments throughout the arch which can be separately treated, and each must evidently be treated in the same manner. We have, for convenience, grouped all the weights together. Hence we have reached another important truth:

In any elastic arch, the closing line of the moment curve, due to the weights alone, must be found from the same conditions and have the same relations to this moment curve that the thrust line has to the curved neutral axis of the arch.

It is to be noticed that the thrust caused by loading an arch induces bending moments of opposite sign from those induced by the loading itself, and that the bending moments really acting on the arch are the difference between those induced by the loading and those induced by the thrust. Hence appears the truth of the following statement, which combines together the separate effects of the thrust and the loading:

If that moment curve, due to the loading, which has for its horizontal thrust the thrust really acting in the arch, be superposed upon the curve of the arch itself, in such a manner that its closing line coincides with the thrust line of the arch, then the bending moment at any point of the arch is equal to the product of the horizontal thrust by the vertical ordinate between the assumed point and the moment curve due to the loading.

Hence the neutral axis of the arch may be considered to play the part of a curved closing line of the moment curve.

This proposition, respecting the coincidence of the thrust line and the closing line, affords the basis for a new graphical investigation of the elastic arch.*

The same principles may be applied to the elastic arch acted on by other than vertical forces.

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*See "New Constructions in Graphical Statics." By Henry T. Eddy, published by D. Van Nostrand, New York, 1877.